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Investigation of Heuristic Implications of Optimal Control Theory

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Abstract—*The main objective of the present work is to study the design aspect of a Quadratic Optimal Regulator for a Spring Mass Dashpot System. In this paper the value of the State Feedback Gain Matrix k is obtained using quadratic optimal regulator. The result of the proposed controller is compared with the controller based on Pole Placement Technique (PPT). It is observed that the optimal control technique outperforms the other controller by finding a proper value of k which contributes towards the stability of the system in a marvelous way.*

Keywords: *Quadratic Optimal Regulator, state feedback gain matrix K, Pole Placement, Spring Mass Dashpot System.*

1. INTRODUCTION

The spring mass damper can be built or represented on the computer instead of going to the workshop to fabricate such system and its performance under various conditions can also be observed without having to subject the real system to these conditions. Hence, you save materials and money, since the system can be used countless times. Energy is also saved because such system is more easily built on a computer than physically. Moreover, it may be very difficult to measure some outputs of some systems such as displacement but such values can be measured with ease through simulation [1]. Springs usually occur physically as a coil of metal, and their idealizations have pretty simple behavior. Compressing the spring will result in the spring pushing back, and stretching the spring will have it trying to pull back towards the start position, so any displacement along the axis of the spring will be countered by an opposite force that will tend to move the spring back to its original position [2].

A continuous system can be modelled either as a discrete- or lumped parameter system with varying number of degrees of freedom or as a continuous system with infinite number of degrees of freedom. The system that is being considered here is of on degree of freedom. The number of independent coordinates needed to describe the configuration of a system at any time during vibration defines the degrees of freedom of the system [1-2].

In recent times optimal control provides the best possible solution to process control problems for a give set of performance objectives. Optimal control and its ramifications have found applications in many different fields, including aerospace, process control robotics, bioengineering, economics, finance, and management science, and it continues to be an active research area within control theory. Optimal control has also found applications in speed control of hybrid electric vehicle [8]. Before the arrival of the digital computer in the 1950s, only fairly simple optimal control problems could be solved. The arrival of the digital computer has enabled the application of optimal control theory and methods to many complex problems. The evolution of optimal control was a significant achievement [9]. The design of control system has a unique aim to meet the desired objectives: specified stability, performance and robustness, by the process of changing a control system's parameters.

In the present work Quadratic Optimal Regulator is proposed for the stability of a spring mass dashpot system used in automobile suspension system. State feedback gain matrix is calculated using MATLAB. Using the value of gain matrix k calculation of step response for the given system is obtained. How the system responds to the initial condition is also obtained. The result of the proposed controller is compared with pole placement technique (PPT) based controller. It is observed that the optimal control technique outperforms the other controller. Organization of the paper is as follows. Section I deals with the introduction of this paper. Modelling part of the spring mass dashpot system is dealt in Section II. Section III discusses the state space controller. Section IV deals with the simulation results and discussion followed by conclusion in the last section.

2. MODELING OF SPRING MASS DASHPOT

System The spring- mass- dashpot system mounted on a Mass less cart is depicted in figure 1.To obtain the mathematical model of this system we assume that the cart is standing still for $t < 0$ and the spring-mass-dashpot system on the cart is also standing still for $t < 0$.In this system, $y(t)$ is the displacement of the cart and is the input to the system. At $t = 0$, the cart is moved at a constant speed, or $\dot{y}$ = constant. The displacement x(t) of the mass is the output. (The displacement is relative to the ground).

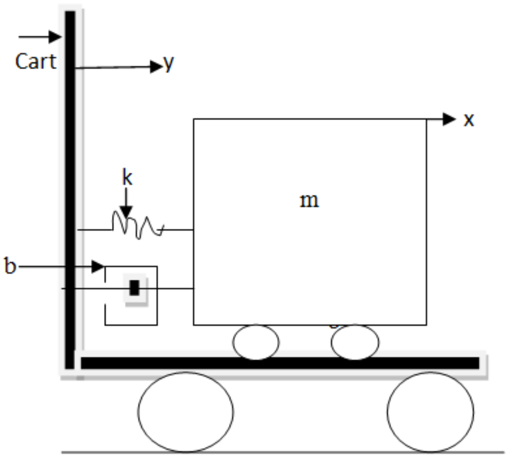

Fig. 1: Generalized block diagram of Spring-mass-dashpot system.

m = Mass of the cart

b = Viscous-friction coefficient

k = Spring constant

We assume that the friction force of the dashpot is proportional to $\dot{x} - \dot{y}$ and that the spring is a linear spring; that is, the spring force is proportional to x – y.

For translational systems, Newton's second law states that

$ma = \sum F$

where m is a mass, a is the acceleration of the mass, and $\sum F$ is the sum of the forces acting on the mass in the direction of the acceleration a. Applying Newton's second law to the present system and noting that the cart is mass less, we obtain :

$$m\frac{d^2x}{dt^2} = -b\left(\frac{dx}{dt} - \frac{dy}{dx}\right) - k(x-y)$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = b\frac{dy}{dt} + ky$$

This equation represents a mathematical model of the system considered above. Taking the Laplace transform of this equation, assuming zero initial condition gives

$$(ms^2 + bs + k)X(s) = (bs + k)Y(s)$$

Taking the ratio, we find the transfer function of the system to be

Transfer Function = $G(s) = \frac{X(s)}{Y(s)} = \frac{bs+k}{ms^2+bs+k}$

Next we obtain a state-space model of this system. We first compare the differential equation for this system

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{b}{m}\dot{y} + \frac{k}{m}y$$

with the standard form

$$\ddot{x} + a_1\dot{x} + a_2x = b_0\ddot{y} + b_1\dot{y} + b_2y$$

Thus on comparing we get

$$a_1 = \frac{b}{m}, a_2 = \frac{k}{m}, b_0 = 0, \quad b_1 = \frac{b}{m}, b_2 = \frac{k}{m}$$

Now to obtain the state space representation we refer to the state space representation of nth-order systems of linear differential equations in which the forcing function involves derivative terms. Thus,

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{b}{m}$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

$$x_1 = x - \beta_0 y = x$$

$$x_2 = \dot{x}_1 - \beta_1 y = \dot{x}_1 - \frac{b}{m}y$$

$$\dot{x}_1 = x_2 + \beta_1 y = x_2 + \frac{b}{m}y$$

$$\dot{x}_2 = -a_2x_1 - a_1x_2 + \beta_2 y = \frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]y$$

and the output equation becomes

$$y = x_1$$

or

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{dy}{dx}\\ \frac{k}{m} - \left(\frac{b}{m}\right)^2\end{bmatrix}y \tag{1}$$

and

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{2}$$

Equations (1) and (2) give a state-space representation of the system. This is not the only state-space representation. There are infinitely many state-space representations for the system.

The variables and constants are taken as :

m = 1 kg, Mass of the cart

k = 2 N\m, Spring Constant

b = 3 Ns\m, Viscous-friction coefficient

Thus now the transfer function obtained is:

$$G(s) = \frac{3s+2}{s^2+3s+2} \tag{3}$$

Thus we obtain the following state-space representation:

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = [3 \quad 2], \quad D = [0] \tag{4}$$

3. STATE SPACE ANALYSIS

State-space approach has often been referred to as modern control design. The power of state variable technique is especially apparent when we need to design the controllers for system having more than one control input or sensed

output. Fig.4 shows the state feedback control system. Here prominently two state space design methods based on pole placement and linear-quadratic regulator (LQR) for optimal control are considered **[**5**].**

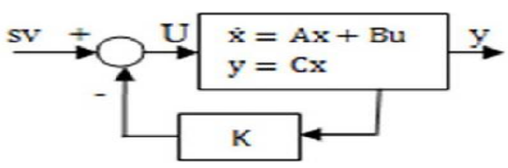

Fig. 2: State feedback control

A. POLE PLACEMENT TECHNIQUE

In pole placement design we place all closed loop poles at desired location. The main goal of a feedback design is to stabilize if it is initially unstable or to improve the relative stability.

Consider the linear time invariant (LTI) system with nth – order state differential equation.

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5}$$

In the state feedback design, the control signal input u is realized as linear combinations of all the states, that is

$$u(t) = -k_1x_1(t) - k_2x_2(t) - \ldots k_nx_n(t) = -kx(t) \tag{6}$$

$$k = [k_1\, k_2 \ldots .\, k_n] \tag{7}$$

K is a constant state feedback gain matrix.

The closed loop system is describe by the state differential equation

$$\dot{x}(t) = (A - Bk)x(t) \tag{8}$$

The characteristics equation of the closed loop system is

$$|sI - (A - Bk)| = 0 \tag{9}$$

The desired characteristics equation is

$$(s - \lambda_1)(s - \lambda_2) \ldots .(s - \lambda_n) = 0 \tag{10}$$

Where λ_1, λ_2….λ_n are desired location of closed loop pole. The selection of desired closed loop poles requires a proper balance of bandwidth, overshoot, sensitivity, control effort etc. The elements of k are obtained by matching the coefficient of (9) and (10). In pole placement design we place all closed loop poles at desired location. The main goal of a feedback design is to stabilize if it is initially unstable or to improve the relative stability. Consider the linear time invariant (LTI) system with nth–order state differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{10}$$

In the state feedback design, the control signal input u is realized as linear combinations of all the states, that is

$$u(t) = -k_1x_1(t) - k_2x_2(t) - k_nx_n(t) = -kx(t) \tag{11}$$

$$k = [k_1\, k_2\, k_n] \tag{12}$$

K is a constant state feedback gain matrix.

The closed loop system is describe by the state differential equation

$$\dot{x}(t) = (A - Bk)x(t) \tag{13}$$

The closed loop system is describe by the state differential equation

$$| sI - (A - Bk)| = 0 \tag{14}$$

The desired characteristics equation is

$$(s - \lambda_1)(s - \lambda_2) \ldots \ldots .(s - \lambda_n) \tag{15}$$

Where $\lambda_1, \lambda_2 \ldots .\lambda_n$ are desired locations of closed loop poles. The selection of desired closed loop poles requires a proper balance of bandwidth, overshoot, sensitivity, control effort etc. The elements of k are obtained by matching the coefficient of (14) and (15).

B. LINEAR QUADRATIC OPTIMAL CONTROLLER

The theory of optimal control is concerned with operating a dynamic system at minimum cost. The case where the system dynamics are described by a set of linear differential equations and the cost is described by a quadratic functional is called the linear quadratic (LQR) problem. One of the main results in the theory is that the solution is provided by the linear-quadratic regulator (LQR).

An advantage of the LQR over the pole placement technique is that the former provides a systematic way of the computing the state feedback control gain matrix. LQR **[**3**]**,**[**4**]** demonstrates excellent performance for designing optimal controller. Here the control objective is to minimize the integral of a quadratic performance index. LQR can be used to design the optimal controller to ensure the optimal regulating and tracking performance [3]. The objective is to find the optimal control law that minimizes the performance index.

$$J = \frac{1}{2}\int_0^\infty (x^TQx + u^TRu)dt \tag{16}$$

Where Q and R are positive semi definite and positive weighting coefficient matrices respectively

The state feedback control low is:

$$u(t) = -kx(t) \tag{17}$$

The solution is [4]

$$k = R^{-1}B^TP \tag{18}$$

Where $k = [k_1\ k_2]$ and P is the solution of Algebraic Riccati equation [4], [9]

$$A^TP + PA - PBR^{-1}B^TP + Q = 0 \tag{19}$$

Where the matrices A and B are the coefficient matrices of the system, from (17) and (18)

$$u(t) = -R^{-1}B^TPx(t) \tag{20}$$

The equation (20) gives the final control law.

4. SIMULATION RESULTS AND DISCUSSION

The system shown above is observed for the response it shows to the initial conditions for different values of state gain matrix k, obtained using pole placement and quadratic optimal controller techniques. Moreover when step response is obtained using both the techniques, it is observed how useful optimal controller proves to be than pole placement. It is clear from equation (4) that the above system is stable for the values of mass, spring constant and viscous coefficient taken.

It is observed that though the system is stable, the stability is improved significantly and the system gives favorable rise time, peak time, settling time, overshoot and steady state error using quadratic optimal controller (LQR).

Figure 3 and 4 shows the step response for the above system for values of gain matrix k using optimal controller and pole placement respectively.

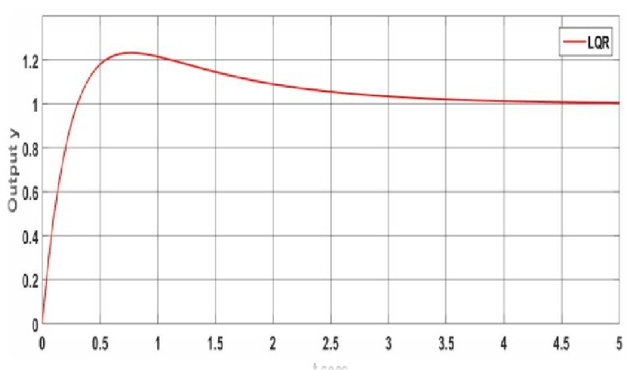

Fig. 3: Step response using LQR

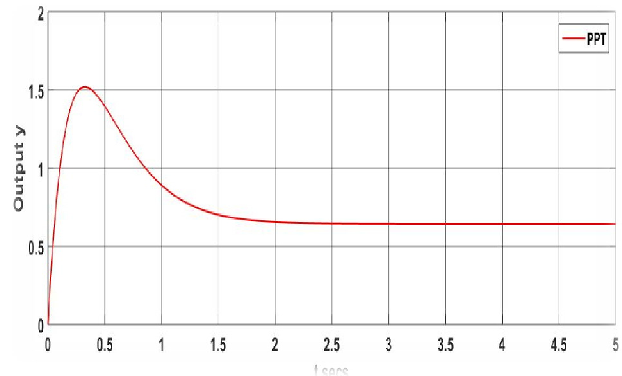

Fig. 4 Step response using PPT

On comparing the response characteristics of LQR and PPT one can easily say by looking at the graphs that response for LQR is far better in comparison to PPT. The table 1 shows the response characteristics as calculated from the MATLAB code [4]:

Table 1

Response Characteristics	LQR	PPT
Rise time	23.1555	4.6062
Settling time	331.0159	190.1034
Settling Min	0.9061	0.6429
Settling Max	1.2321	1.5186
Overshoot	22.6567	136.2219
Undershoot	0	0
Peak	1.2321	1.5186
Peak time	78	34

It is clear that the maximum overshoot for LQR being 22.6567 is very less as compared to PPT having an overshoot of 136.2219. Thus response of LQR is less oscillatory and desirable. The Peak value and Settling Max time for PPT is more in comparison to LQR. Thus LQR depicts a response that is completely satisfactory and the system tracks the input also and achieves stability at the final value i.e. 1. In PPT the output is not settled at the final value and settles close to the final value. So clearly LQR proves to be far better in comparison to PPT and outperforms it.

Overshoot in the system implies something with the energy consumption. The presence of overshoot is similar to the uneven road where vehicle would consume more fuel hence this way optimal controller extenuates the overshoot and gives less oscillatory response which in turn consumes less energy. Now there are two kinds of control objective, servo problem and regulatory problem. For servo problem, we take set value (SV) as unit step signal and for our regulatory problem we take it as zero and disturbance signal of any kind need to be regulated efficiently. For regulatory problem also the LQR controller displays better performance than PPT based controller.

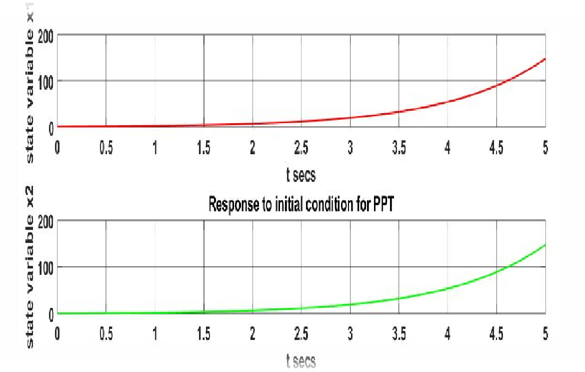

Fig. 5 Response to initial condition using PPT

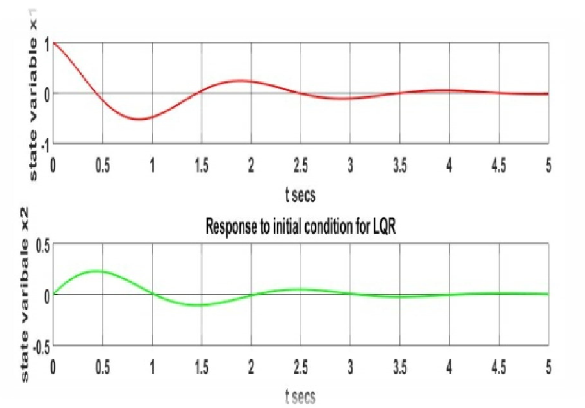

Fig. 6 Response to initial condition using LQR

Figure 5 and figure 6 show the response to the initial condition for PPT and LQR respectively.

The initial condition considered is:

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (21)$$

It is clear from figure 5 that graphs for state variable x1 and x2 are similar as the value of k obtained from the pole placement technique is same i.e. k=[2 2]. On the other hand the value of state feedback gain matrix for optimal controller is k=[4.5 12] i.e. different and thus the graphs for state variables x1 and x2 are different accordingly.

5. CONCLUSION

In this work a spring mass dashpot system is proposed. Here Quadratic Optimal controller's (LQR) design aspect based on state feedback gain matrix is investigated along with Pole Placement (PPT) to see that using which technique the stability of the system is improved. In this paper LQR for analysis of heuristic implications of optimal control is proposed and is compared with PPT technique. Also the response to initial conditions is presented for both the techniques. Looking at the step response characteristics we observe that LQR has much less overshoot compared to PPT and hence provides less oscillatory response with better stability. System tracks the input better using LQR whereas using PPT it settles but fails to track input properly. Settling max for LQR is less. Thus it is observed that optimal control proves to be a better choice and improves system stability in an amazing manner.

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